J. Fluid Mech. (1991), *vol.* 230, *pp.* 355–364
Printed in Great Britain

Plane Stokes flow driven by capillarity on a free surface. Part 2. Further developments

By ROBERT W. HOPPER
Chemistry and Materials Science Department, Lawrence Livermore National Laboratory, Livermore, CA, USA

(Received 16 October 1990)

For the free creeping viscous incompressible plane flow of a finite region, bounded by a simple smooth closed curve and driven solely by surface tension, analyzed previously, the shape evolution was described in terms of a time-dependent mapping function $z = \Omega(\zeta, t)$ of the unit circle, conformal on $|\zeta| \leqslant 1$. An equation giving the time evolution of the map, typically in parametric form, was derived. In this article, the flow of the infinite region exterior to a hypotrochoid is given. This includes the elliptic hole, which shrinks at a constant rate with a constant aspect ratio. The theory is extended to a class of semi-infinite regions, mapped from $\operatorname{Im}\zeta \leqslant 0$, and used to solve the flow in a half-space bounded by a certain groove. The depth of the groove ultimately decays inversely with time.

1. Introduction

A theory was developed previously (Hopper 1990, hereinafter referred to as Part 1) for a special type of moving free boundary problem in fluid dynamics: briefly, creeping viscous incompressible plane flow in a finite region, bounded by a simple smooth closed curve and driven slowly by surface tension. Such problems are self-contained in that the applied tractions are intrinsic in the geometry. The objective is to determine exactly the time evolution of the shape. The problems are nonlinear owing to the large changes in shape. The region in the complex z-plane is described in terms of a time-dependent conformal mapping function $\Omega(\zeta, t)$ on the fixed region $|\zeta| \leqslant 1$ of the complex ζ-plane. An equation giving the time evolution of $\Omega(\zeta, t)$ was derived. In practice, it has been necessary to conjecture a parametric form, $\Omega[\zeta; a_1(t), a_2(t), \ldots]$. The correctness of a candidate map is verified, and the time dependence of the parameters determined, using the theory. When the conjectured form holds and the equations can be solved, the evolution of the shape with time is obtained in simple, exact and essentially closed form. Several examples were given. References to prior related work are cited in Part 1. In the present article, the theory is extended and further examples are given.

Let us review the main points developed in Part 1. The region is regarded as the cross-section of an infinitely long isothermal general cylinder of Newtonian viscous liquid having dynamic viscosity η, density ρ and surface tension γ, in a gravitational field $\boldsymbol{g}$, all these being constants. Let R_0 be a characteristic distance of the problem. A normalization scheme appropriate to creeping flow driven by surface tension is used. Denoting the position $\boldsymbol{x}_0$ and the time t_0, the corresponding dimensionless variables are $\boldsymbol{x} = \boldsymbol{x}_0/R_0$ and $t = \gamma t_0/\eta R_0$. In the limit where the Suratman number $(\rho\gamma R_0/\eta^2)$ and the Bond number $(\rho g R_0^2/\gamma)$ approach zero, inertial and gravitational effects become negligible in comparison with capillarity and viscous ones, and the

Navier–Stokes momentum equation in dimensionless form reduces to that of Stokes. It may not be always possible to assure that inertial effects remain insignificant for certain infinite regions. The boundary condition is that the surface traction is in the outward normal direction and equals the negative of the curvature. The applicability of the analysis to real liquids is subject to other limitations discussed in Part 1. Ignoring these concerns, we take Stokes equations for plane flow together with this classical boundary condition in dimensionless forms as the starting point. The theory employs the Kolostoff–Muskhelishvili equations of elasticity theory (Muskhelishvili 1953; Sokolnikov 1956), where the stresses and displacements (here, the velocity) are expressed in terms of two analytic functions ϕ and ψ.

Let $z = \Omega(\zeta, t)$ conformally map $|\zeta| \leqslant 1$ onto the (dimensionless) region of interest. Rigid-body translations and rotations may appear, depending on how Ω is chosen. These are not physically significant and are ignored. Primes denote complex derivatives with respect to the independent (complex) variable; an overdot, the derivative with respect to time; and an asterisk, the complex conjugate. Let $\sigma = e^{i\vartheta}$. Thus, ζ denotes general points in the ζ-plane, while σ always indicates points on the unit circle of the ζ-plane. The surface tractions are intrinsic in the map $\Omega(\zeta, t)$. The requirement that the surface velocity be such that a fluid element at $z = \Omega(\sigma, t)$ move to $z + dz = \Omega(\sigma + d\sigma, t + dt)$ in the time increment dt is sufficient to determine $\phi(\zeta, t)$ in terms of $\Omega(\zeta, t)$:

$$\phi(\zeta) = -\zeta\Omega'(\zeta)\,F(\zeta) + \dot{\Omega}(\zeta), \tag{1}$$

where

$$F(\zeta, t) \equiv \frac{1}{2\pi i}\oint \frac{1}{2|\Omega'(\sigma, t)|}\frac{\sigma+\zeta}{\sigma(\sigma-\zeta)}\,d\sigma. \tag{2}$$

Substituting into the surface-traction boundary condition, it is found that the values of $\psi(\zeta, t)$ on the boundary are given explicitly by

$$\psi(\sigma) = \Omega(\sigma)^*\left\{\left[1+\frac{\sigma\Omega''(\sigma)}{\Omega'(\sigma)}\right]F(\sigma)+\sigma F(\sigma)-\frac{\dot{\Omega}(\sigma)}{\Omega'(\sigma)}\right\}-[\sigma\Omega'(\sigma)]^*F(\sigma)-\Omega(\sigma)^*. \tag{3}$$

The boundary values are defined to be the limit as the boundary point is approached from within $|\zeta| < 1$. The validity of the chosen mapping, and the time dependence itself, are determined by the requirement that $\psi(\sigma, t)$ be the boundary value of a function analytic throughout $|\zeta| \leqslant 1$ for $t > 0$; that is, that the function on the unit circle can be continued analytically onto the unit disk. Some conjectures regarding important forms of Ω were argued in Part 1; they serve as guides for constructing appropriate maps. It is implicit in (3) that the area of the liquid is unchanging.

2. Hypotrochoidal holes (including elliptic)

These equations carry over to maps of the infinite region exterior to a smooth simple curve. The unit disc is mapped by $z = \omega_N(\zeta) = \zeta^{-1}[1 + \zeta^N/N\,1)]$ onto the region exterior to an N-cusped regular hypocycloid having one of its cusps on the positive real axis. Smaller discs $|\zeta| < \text{const} < 1$ are mapped onto the exteriors of hypotrochoids. Hypotrochoidal holes may then treated in the same manner as were epitrochoidal bodies (Part 1). The $N = 2$ case gives the elliptic hole. We impose the condition that the stress components vanish as $|z| \to \infty$. (If the region is regarded as the infinite limit of a large body, then there can be no externally applied pressure.)

From the reasoning of Part 1, it may be conjectured that

$$\Omega_N(\zeta, t) = \zeta^{-1}[a_N(t) + b_N(t)\,\zeta^N], \quad 0 \leqslant |\zeta| \leqslant 1,\ N \geqslant 2; \tag{4}$$

a_N and b_N are real and positive, and the subscripts N will be omitted when no confusion results. The positive direction along the boundary in z-space is clockwise around the hole. The elliptic hole ($N=2$) is in some respects exceptional, but most of the analysis applies generally to $N \geqslant 2$. The analysis is done on the normalized problem. The area of the hole will change with time; the constancy of the material area (continuity) is assured by the equations of the theory (see Part 1) and will not be analysed separately.

Equations (1)–(3) again hold provided the positive sense of a contour always has the region on the left. From (4).

$$|\Omega'(\sigma)| = [a^2 - 2ab(N-1)\cos N\vartheta + (N-1)^2 b^2]^{\frac{1}{2}}. \tag{5}$$

This is the same form that occurs in §4.1 of Part 1, where it was shown that

$$F^{(n)}(0) = 0 \quad \text{for} \quad 1 \leqslant n < N, \tag{6a}$$

$$F(0) = (1/\pi a)K[(N-1)\,b/a], \tag{6b}$$

where $K(k)$ is the complete elliptic integral of the first kind. Because the stress has been required to vanish as $|z| \to \infty, \phi(\zeta)$ and $\psi(\zeta)$ must be analytic throughout $|\zeta| \leqslant 1$ (see Sokolnikov 1956, equation 76.11). From (2),

$$\lim_{\zeta\to}\left\{\frac{a-(N-1)\,b\zeta^N}{\zeta}[F(0)+F'(0)\zeta+\ldots]+\frac{a+b\zeta^N}{\zeta}\right\} = \phi(0) \quad (\neq \infty), \tag{7}$$

which requires
$$a + aF(0) = 0. \tag{8}$$

Equation (3) becomes

$$\psi(\sigma) = \left(\frac{a+b\sigma^N}{\sigma}\right)^*\left[-\frac{a+(N-1)^2 b\sigma^N}{a-(N-1)\,b\sigma^N}F(\sigma)+\sigma F'(\sigma)-\frac{\dot{a}-(N-1)\,\dot{b}\sigma^N}{a-(N-1)\,b\sigma^N}\right]$$
$$+\left[\frac{a-(N-1)\,b\sigma^N}{\sigma}\right]^* F(\sigma)-\left(\frac{\dot{a}+\dot{b}\sigma^N}{\sigma}\right)^* \tag{9a}$$

$$= \sigma^{-(N-1)}\{(a\sigma^N+b)[\ldots]+[a\sigma^N-(N-1)\,b]F(\sigma)-(\dot{a}\sigma^N+\dot{b})\}. \tag{9b}$$

The analytic continuation giving $\psi(\zeta)$ is obtained by replacing σ with ζ. Expanding $F(\zeta)$ in (9b) as needed, analyticity of $\psi(\zeta)$ on $|\zeta| \leqslant 1$ requires

$$b[-F(0)-\dot{a}/a]-(N-1)\,bF(0)-\dot{b} = 0, \tag{10}$$

which with (8) gives the second condition,

$$\dot{b}+(N-1)\,bF(0) = 0. \tag{11}$$

Combining (4) and (11), and integrating:

$$b_N(t) = C_N\,a_N(t)^{N-1}, \quad N \geqslant 2 \tag{12}$$

Equation (12) is general. The special feature of the elliptic hole is that

$$b_2(t) = C_2\,a_2(t). \tag{13}$$

This means that any elliptic hole shrinks without change of the axial ratio. Letting m be the ratio of the minor axis to the major, $C_2 = (1-m)/(1+m)$. Combining with (6b) and (8), and integrating:

$$a_2(t) = a_2(0) - K[(1-m)/(1+m]t/\pi. \tag{14}$$

For $N > 2$, the shapes change and it is natural to take the time zero as the singular (i.e. cusped) limit. This is useless for $N = 2$: there is no natural time zero. For the ellipse, then, the choice of the characteristic distance R_0 should be chosen as a characteristic distance at the initial time, which may arbitrarily be taken as zero. (The solution will then be valid for negative as well as positive times.)

The behaviour in dimensional form may be summarized simply: the (dimensional) semi-minor axis is given by

$$L_{\text{s-min}}(t_0) = L_{\text{s-min}}(0) - [2m/\pi(1+m)]K[(1\text{-}m)/(1=m)](\gamma t_0/\eta). \tag{15}$$

Equation (15) together with the fact that m is constant describes the flow of the infinite region with an elliptic hole. Of course, the semi-major axis ($L_{\text{s-maj}} = L_{\text{s-min}}/m$) shrinks at a constant rate $1/m$ of that of the semi-minor. As the axial ratio $m \to 0$, $\mathrm{d}L_{\text{s-min}}(t)/\mathrm{d}t \to 0$ and $\mathrm{d}L_{\text{s-maj}}(t)/\mathrm{d}t \to -\infty$. This limiting behaviour is approximated by

$$L_{\text{s-min}}(t) \approx L_{\text{s-min}}(0) - (m/\pi[\ln(4/m)](\gamma t_0/\eta) \quad \text{as} \quad m \to 0. \tag{16}$$

The elementary result for the shrinking of a circular hole of (dimensional) radius r_0 is recovered from (15) when $m = 1$:

$$r_0(t) = r_0(0) - (\gamma t_0/2\eta). \tag{17}$$

Return now to (12) and consider the $N > 2$ holes. The area of the hole will certainly change. It is convenient to normalize the problem such that πR_0^2 is the hole area of the singular limit. Taking this as the time zero and noting when zeros of $\Omega'(\sigma)$ first occur, $b_N(0) = a_N(0)/(N-1)$. Equation (24b) of Part 1 gives the hole area, from which one obtains

$$a_N(0) = [(N-1)/(N-2)]^{\frac{1}{2}}, \quad b_N(0) = [(N-1)/(N-2)^{-\frac{1}{2}} \quad (N > 2). \tag{18a, b}$$

Using these in (12) gives C_N, whence

$$b_N(t) = (N-2)^{-1}[(N-2)/(N-1)]^{N/2} a_N(t)^{N-1} \quad (N > 2). \tag{19}$$

Using (6b), (8) and (19), and integrating:

$$t = \pi \int_{a_N(t)}^{a_N(0)} K\{[(N-2)/(N-1)]^{(N-2)/2} a_N^{N-2}\}^{-1}\, \mathrm{d}a_N \quad (N > 2) \tag{20}$$

All this can be recast in a more convenient form by changing variables in (20) with $k = [N-2)/(N-1)]^{(N-2)/2} a_N^{N-2}$, and defining $\nu = [a_N(t)/a_N(0)]^{N-2}$. Then (4) and (20) become

$$\Omega_N[\sigma, \psi(t)] = \left(\left(\frac{N-1}{N-2}\right)^{\frac{1}{2}} \nu^{1/(N-2)}\, \zeta^{-1}\right)\left[1 + \frac{\nu\zeta^N}{N-1}\right] \quad (N > 2), \tag{21}$$

$$t(\nu) = \frac{\pi}{N-2}\left(\frac{N-1}{N-2}\right)^{\frac{1}{2}} \int_{\nu}^{1} [k^{1-1/(N-2)} K(k)]^{-1}\, \mathrm{d}k \quad (N > 2). \tag{22}$$

Equations (21) and (22) describe the closing of any $N > 2$ hypotrochoidal hole, where the characteristic length R_0 of the normalization is such that the area of the hole at $t = 0$ is πR_0^2. The astroid family ($N = 4$) is shown in figure 1.

Because the integrand of (22) as $k \to 0$ is $\sim (2)/\pi k^{-1+1/(N-2)}$, it is obvious that the hole closes in a finite time. As $\nu \to 0$ the hole becomes (for $N > 2$) a tiny circle of radius

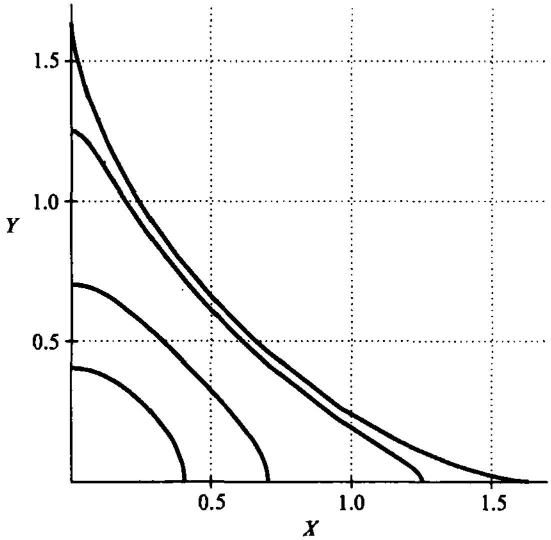

FIGURE 1. Hypotrochoidal holes for $N = 4$, showing one quadrant for times $t = 0$, 0.300, 1.000 and 1.500 corresponding to $\nu = 1$, 0.691, 0.276 and 0.102, respectively. The full figure is symmetric across the axes. The singular case ($t = 0$) is the $N = 4$ hypocycloid (the 'astroid'.

$[(N-1)/(N-2)]^{\frac{1}{2}}\nu^{1/(N-2)}$ that closes at a speed of $-\frac{1}{2}$, in accord with (17). As $N \rightarrow \infty$, the singular limit becomes a circular hole of unit radius with tiny cycloidal ripples. The ripples damp quickly leaving a circular hole shrinking, again, at a speed of $-\frac{1}{2}$. This case can be renormalized to obtain the levelling of half-space bounded by a trochoid and yields the same result as obtained from epitrochoidal bodies (Part 1).

3. Theory for half-space regions

Half-space regions have previously been treated as limits of finite regions (Part 1). In general, however, it is useful to have available a formalism specifically tailored for half-space problems. In this section, such a formalism is developed for a class of simply connected half-space regions. It seems natural to employ a mapping from the upper half-plane, $z = \Omega(\zeta, t)$, $\operatorname{Im}\zeta \geqslant 0$. The region is oriented such that the fluid extends to infinity in the upper half of the z-plane and is restricted to be level at infinity (i.e. $\operatorname{Im}\Omega(\xi+\mathrm{i}0, t) \rightarrow 0$ as $|\xi| \rightarrow \infty$). Maps not meeting this requirement involve branch points, leading to fundamental difficulties (Part 1).

The derivation is very similar to that of Part 1 for finite regions. Let $\zeta = \xi = \mathrm{i}\eta$. The description of the flow is that the surface velocity $u(\xi, t)$ be such that a fluid element at $z = \Omega(\xi, t)$ move to $z = \mathrm{d}z = \Omega(\xi+\mathrm{d}\xi, t+\mathrm{d}t)$ in the time increment $\mathrm{d}t$. Let $\mathrm{d}\xi = \dot{\Xi}(\xi, t)\,\mathrm{d}t$, where $\dot{\Xi}(\xi, t)$ is real. Then

$$u(\xi, t) = \Omega'(\xi, t)\,\dot{\Xi}(\xi, t) + \dot{\Omega}(\xi, t). \tag{23}$$

The Kolosoff–Muskhelishvili velocity and traction equations are

$$\phi(\xi, t) - \frac{\Omega(\xi, t)}{\Omega'(\xi, t)^*}\phi'(\xi, t)^* - \psi(\xi, t)^* = 2u(\xi, t), \tag{24}$$

$$\phi(\xi, t) + \frac{\Omega(\xi, t)}{\Omega'(\xi, t)^*}\phi'(\xi, t)^* + \psi(\xi, t)^* = -\mathrm{e}^{\mathrm{i}\alpha(\xi, t)} = \mathrm{i}\frac{\Omega'(\xi, t)}{|\Omega'(\xi, t)|}. \tag{25}$$

In (25), $\alpha(\xi,t)$ is the angle between the outward normal and the Re z-axis, and an arbitrary constant has been set to zero. Combining:

$$\phi(\xi,t) = \mathrm{i}\Omega'(\xi,t)\left[\frac{1}{2|\Omega'(\xi,t)|} - \mathrm{i}\dot{\Xi}(\xi,t)\right] + \dot{\Omega}(\xi,t) \tag{26}$$

The terms in square brackets are the real and imaginary parts of the boundary values of a function analytic on $\operatorname{Im}\zeta \geqslant 0$, and this function is

$$G(\zeta,t) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{1}{|\Omega'(\xi,t)|}\frac{1}{\xi-\zeta}\mathrm{d}\xi, \quad \operatorname{Im}\zeta > 0, \tag{27}$$

with boundary values given by the Plemelj formula

$$G(\xi_0,t) \equiv \lim_{\zeta\to\xi_0} G(\zeta,t) = \frac{1}{2|\Omega'(\xi_0,t)|} + \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{1}{|\Omega'(\xi,t)|}\frac{1}{\xi-\xi_0}\mathrm{d}\xi, \tag{28}$$

where the principal value of the integral is implied. The validity of (28) requires that the following Hölder condition for the 'neighbourhood of the point at infinity' (see Muskhelishvili 1953, §71) be satisfied: for sufficiently large ξ_1, with $\xi_1 < \xi_2$,

$$\left|\frac{1}{|\Omega'(\xi_2,t)|} - \frac{1}{|\Omega'(\xi_1,t)|}\right| \leqslant A\left|\frac{1}{\xi_2} - \frac{1}{\xi_1}\right|^{\nu}, \quad 0 < \nu \leqslant 1. \tag{29}$$

This implies that $|\Omega'(\xi,t)| \to |\Omega'(\infty,t(| + O(|\xi|^{-m}), \mu > 0$, as $|\xi| \to \infty$.

Assuming that all this applies, the analytic continuation of (26) onto $\operatorname{Im}\zeta \geqslant 0$ gives an explicit expression for $\phi(\zeta,t)$ on $\operatorname{Im}\zeta \geqslant 0$. Upon differentiating, inserting into the traction equation, and noting that $G(\xi,t)^* = |\Omega'(\xi,t)|^{-1} - G(\xi,t)$, one obtains

$$\mathrm{i}\psi(\xi,t) = \Omega(\xi,t)^*\left[\frac{\Omega''(\xi,t)}{\Omega'(\xi,t)}G(\xi,t) + G'(\xi,t) - \mathrm{i}\frac{\dot{\Omega}'(\xi,t)}{\Omega'(\xi,t)}\right] + \Omega'(\xi,t)^* G(\xi,t) - \mathrm{i}\dot{\Omega}(\xi,t)^*. \tag{30a}$$

More compactly,

$$\mathrm{i}\Omega'(\xi,t)\,\psi(\xi,t) = \frac{\mathrm{d}}{\mathrm{d}\xi}[\Omega'(\xi,t)\,\Omega(\xi,t)^* G(\xi,t)] - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}[\Omega'(\xi,t)\,\Omega(\xi,t)^*]. \tag{30b}$$

As with (3), the validity of the chosen mapping and its time-dependence are determined by the requirement that $\psi(\xi,t)$ be the boundary value of a function analytic on $\operatorname{Im}\zeta \geqslant 0$ for $t > 0$; that is, that the function on the real line can be continued analytically onto the upper half-plane. Conservation of the fluid area requires that $\Omega'(\zeta,t) \to$ const (independent of t) as $|\zeta| \to \infty$. The presence of branch points in $\Omega(\zeta,t)$ creates the same sort of problems as discussed in Part 1.

4. Isolated groove on half-space

The flattening of half-space bounded initially by a certain isolated groove may be solved using the theory of §3. The candidate map was deduced by using a linear fractional transformation to expand the lobe of the $N = 1$ epitroichoidal body (Part 1) to fill a half-plane in z-space. A remnant groove remains. The disk $|\zeta| \leqslant 1$ was mapped into a half-plane of ζ-space. The condition $\Omega'(\infty,t) = \text{const} = 1$ then leads to

$$\Omega(\zeta,t) = \zeta - \frac{\lambda(t)^2}{\zeta + \mathrm{i}\mu(t)}, \quad \operatorname{Im}\zeta \geqslant 0. \tag{31}$$

The fluid lies in the upper half-plane. The critical points of the map are a pole of $\Omega(\xi, t)$ at $\zeta = -\mathrm{i}\mu$, and zeros of $\Omega'(\zeta, t)$ at $\zeta = -\mathrm{i}(\mu \pm \lambda)$, so conformality requires $\mu(t) > \lambda(t) > 0$ for $t > 0$. The hypotheses of §3 are met. The singular initial condition, for which the groove is sharp and of unit depth, is given by $\lambda(0) = \mu(0) = 1$. This curve is known as a pedal of a parabola. The flow is determined by adjusting the parameters $\lambda(t)$ and $\mu(t)$ so that $\psi(\zeta, t)$, given by the analytic continuation of (30*a*), is analytic throughout $\operatorname{Im}\zeta \geqslant 0$ for $t > 0$.

Substituting (31) into (30*a*) and noting that $\Omega(\xi, t)^* = \xi - \lambda^2/(\xi - \mathrm{i}\mu)$, etc. it is obvious that the resulting expression for $\psi(\xi, t)$ continues analytically onto $\operatorname{Im}\zeta \geqslant 0$ except at the point $\zeta = \mathrm{i}\mu$. Letting $\zeta = \mathrm{i}\mu + \epsilon$ and using Taylor expansions,

$$\mathrm{i}\psi(\mathrm{i}\mu+\epsilon) = -\frac{\lambda^2}{\epsilon}\left\{\frac{\Omega''(\mathrm{i}\mu)}{\Omega'(\mathrm{i}\mu)}[G(\mathrm{i}\mu)+\dot{\mu}] + G'(\mathrm{i}\mu) + \mathrm{i}2\frac{\dot{\lambda}}{\lambda}\left[\frac{1}{\Omega'(\mathrm{i}\mu)} - 2\right]\right\}$$

$$+\frac{\lambda^2}{\epsilon^2}[G(\mathrm{i}\mu)+\epsilon G'(\mathrm{i}\mu)-\dot{\mu}] + O(\epsilon^0). \quad (32)$$

Avoiding the poles of first and second order at $\zeta = \mathrm{i}\mu$ then requires

$$\dot{\mu} = G(\mathrm{i}\mu), \quad \dot{\lambda} = -\frac{\lambda^3/4\mu^3}{1-\lambda^2/2\mu^2}G(\mathrm{i}\mu). \quad (33a, b)$$

Divide the latter by the former, let $\nu = \lambda/\mu$, integrate using partial fractions, and use $\lambda(0) = \mu(0) = 1$. Then

$$\mu = 3^{\frac{1}{2}}\nu^{-1}(4-\nu^2)^{-\frac{1}{2}}. \quad (34)$$

The map (31) may now be rewritten:

$$\Omega(\zeta, t) = \zeta - \mu^2\nu^2/(\zeta + \mathrm{i}\mu). \quad (35)$$

The time-dependence is obtained from (33*a*). Using $\Omega'(\xi; \lambda, \mu)$ in (27), at $\zeta = \mathrm{i}\mu$, and using equations 3.152–1 and 8.126–3 of Gradshteyn & Rhyzhik (1980), one easily obtains

$$G(\mathrm{i}\mu; \lambda, \mu) = \frac{\mu}{\pi}\int_0^\infty \{[\xi^2+(\mu+\lambda)^2][\xi^2+(\mu-\lambda)^2]\}^{-\frac{1}{2}}\,\mathrm{d}\xi$$

$$= \frac{\mu}{\pi}\frac{1}{\lambda+\mu}K\left(\frac{2(\lambda\mu)^{\frac{1}{2}}}{\lambda+\mu}\right) = \frac{1}{\pi}\frac{1}{1+\nu}K\left(\frac{2\nu^{\frac{1}{2}}}{1+\nu}\right) = \frac{1}{\pi}K(\nu). \quad (36)$$

From (33) and (35),

$$\frac{\mathrm{d}\nu}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\lambda}{\mu}\right) = \frac{1}{\mu}(\dot{\lambda}-\nu\dot{\mu}) = -\frac{1}{\mu}\left(\frac{\nu^3}{4-2\nu^2}+\nu\right)G(\mathrm{i}\mu) = -\frac{1}{2\sqrt{3^{\frac{1}{2}}}\pi}\frac{\nu^2(4-\nu^2)^{\frac{3}{2}}}{(2-\nu^2)}K(\nu). \quad (37)$$

Noting that $\nu(0) = 1$ and integrating:

$$t(\nu)2\sqrt{3}\pi\int_\nu^1 (2-k^2)[k^2(4-k^2)^{\frac{3}{2}}K(k)]^{-1}\,\mathrm{d}k. \quad (38)$$

Expanding the integrand as $k \rightarrow 0$ provides the long-time behaviour:

$$t(\nu) \sim t(\nu_0) + 2(6)^{\frac{1}{2}}[(\nu^{-1}-\nu_0^{-1}) - \tfrac{3}{8}(\nu-\nu_0)] \quad (0 < \nu < \nu_0,\ \nu_0 \rightarrow 0) \quad (39a)$$

$$t(\nu) \approx 4.899(\nu^{-1}+\nu) - 65.9 \quad (\pm 0.1 \quad \text{for} \quad \nu < \nu_0 = 0.05). \quad (39b)$$

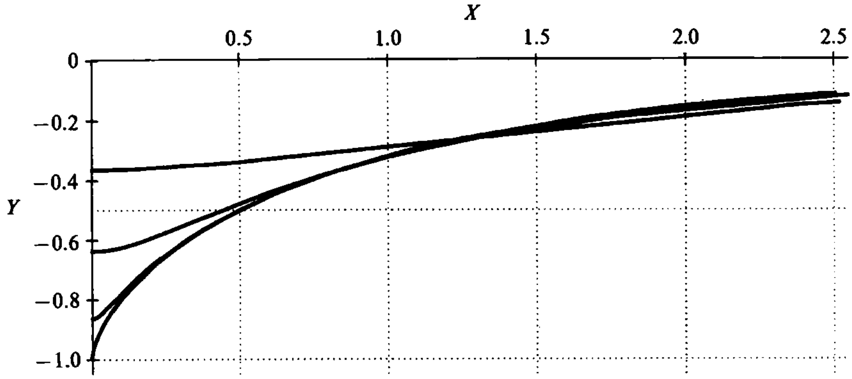

FIGURE 2. Flattening of an isolated groove, showing the right half of the shape for times $t = 0$, 0.100, 0.500 and 2.000 corresponding to $\nu = 1$, 0.894, 0.691 and 0.412, respectively. The singular case ($t = 0$) is the pedal of a parabola.

Equations (35), (35) and (38) constitute the solution. Inverting the figure for aesthetical reasons, the boundary is given parametrically by

$$X(\xi, t) = \xi[1-\mu^2\nu^2/(\xi^2+\mu^2)], \tag{40a}$$

$$Y(\xi, t) = \mu^3\nu^2/(\xi^2+\mu^2). \tag{40b}$$

The shape evolution is depicted in figure 2. The depth of the groove is $-Y(0,t) = \mu\nu^2$, and the curvature at its bottom is $\kappa(0,t) = 2\nu^2[\mu(1-\nu^2)^2]^{-1}$.

From (34) and (39a), the depth decays as $\mu\nu^2 \sim 3(2)^{\frac{1}{2}}t^{-1}$ as $t \rightarrow \infty$, which is much slower than the exponential decay of the trochoid obtained in Part 1. The difference is because a disturbance that initially is localized spreads out with time, whereas a periodic one cannot; so the curvatures driving the flow decay more rapidly for the former. The t^{-1} decay accords with a fairly general conclusion of Kuiken (1990), §4. The motion of the critical points of the map with time is of interest. The pole at $\zeta = -\mathrm{i}\mu \rightarrow -\mathrm{i}\infty$ as $t \rightarrow \infty$. The zeros of $\Omega'(\zeta)$ at $\zeta = -\mathrm{i}(\mu \pm \lambda)$ move with it, ultimately being at a fixed distance from it: $\lambda(\infty) = 3^{\frac{1}{2}}/2$.

5. Further developments and discussion

A well-known device (Sokolnikov 1956, §80) permits testing the hypothesis that the interior of an ellipse circularizes through a sequence of ellipses. Let

$$\Omega(\zeta) = (a+b\zeta^2)/\zeta, \quad 1 \leqslant |\zeta| \leqslant (a/b)^{\frac{1}{2}}, \quad 0 < (b/a)^{\frac{1}{2}} < 1. \tag{41}$$

This maps $|\zeta| = 1$ onto an ellipse having semi-axes $(a+b)$ and $(a-b)$, and $|\zeta| = (a/b)^{\frac{1}{2}}$ onto the line segment $[|\mathrm{Re}\, z| \leqslant 2(ab)^{\frac{1}{2}}, \mathrm{Im}\, z = 0]$ (the 'cut'). The conditions on $|\zeta| = 1$ lead again to (1)–(3), but now $\psi(\sigma)$ need be continued analytically only onto the annulus $1 \leqslant |\zeta| \leqslant (a/b)^{\frac{1}{2}}$. The conventional directional senses of the contours are used: $F(\zeta)$ is defined by (1) but with the integration in the clockwise sense (since this gives a counter-clockwise direction in the z-plane), and the domain is $|\zeta| \geqslant 1$, throughout which $F(\zeta)$ is analytic. Further conditions arise, however, including the necessity that ϕ be continuous in the z-space region, including on the cut: otherwise, unacceptable discontinuities of the velocity and stress fields occur. This implies that $\phi[(a/b)^{\frac{1}{2}}\sigma] = \phi[(a/b)^{\frac{1}{2}}\sigma^*]$. By symmetry, $\phi(\zeta^*) = \phi(\zeta)^*$, so this condition requires

$\operatorname{Im}\phi[(a/b)^{\frac{1}{2}}\sigma] = 0$. A constant area implies $a\dot{a} = b\dot{b}$. Together, these are found to imply

$$\operatorname{Re} F[(a/b)^{\frac{1}{2}}\sigma] = \tfrac{1}{2}[(a/b)^2 - 1](\dot{a}/a). \tag{42}$$

$F(\zeta)$ is evaluated by integrating around a closed contour consisting of the unit circle and lines along the branch cuts $[|\operatorname{Re}\zeta| \leqslant (b/a)^{\frac{1}{2}}, \operatorname{Im}\zeta = 0]$. It is found that

$$\operatorname{Re} F[(a/b)^{\frac{1}{2}}\sigma] = \frac{1}{\pi}\int_0^1 \frac{1}{(1-\xi^2)^{\frac{1}{2}}(1-(b^2/a^2)\xi^2)^{\frac{1}{2}}} \frac{1-(b\xi/a)^4}{1+(b\xi/a)^4 - 2(b\xi/a)^2\cos 2\vartheta}\,\mathrm{d}\xi. \tag{43}$$

This is manifestly not independent of $\sigma(=\mathrm{e}^{\mathrm{i}\vartheta})$, conflicting with (42). Therefore, the original hypothesis was false, and a body bounded initially by an ellipse does not become circular through a sequence of ellipses.

The formal theory is readily extended to doubly-connected finite regions mapped from the annulus $0 < \rho(t) \leqslant |\zeta| \leqslant 1$, using the same conceptual scheme. Separate expressions, similar to (1)–(3), are found to apply on the two boundaries. The requirement is then that the boundary values $\psi(\sigma)$ and $\psi(\rho\sigma)$ can be continued onto the ζ-annulus such that not only are there no singularities but also the *same* analytic function $\psi(\zeta)$ results. There is also separate a condition arising from $\phi(\zeta)$. The formalism readily yields the known solution to the elementary problem of the closure of a concentric circular annulus. To date, however, the author has been unable to guess a map that corresponds to a non-trivial flow. Publication of the theory therefore seems unwarranted at this time, though it is available in report form (Hopper 1991). A physically interesting case was the conjecture that a ring bounded by confocal ellipses collapses through a sequence of shapes bounded by other pairs of confocal ellipses. It proved impossible to satisfy the ϕ-condition, so the conjecture is false (Hopper 1991).

The flow exterior to a parabola has also been analyzed, by a direct method. It is found that, in the absence of externally imposed stresses, the parabolic boundary simply translates without change of shape. The analysis, together with a treatment of the effects of certain fields imposed at infinity, will be published elsewhere.

The theory developed in these articles has yielded the evolution of shapes whose singular ($t = 0$) limit involves cusps directed into the liquid. No time-dependent map has been discovered giving the flow of a shape whose singular limit has a corner or outward-pointing cusp. Such maps must exist, but the simple ones that first come to mind (Part 1) prove incorrect – for fairly general mathematical reasons. Intuitively, these conjectured maps may also seem physically implausible. Consider, for example, a long slender oval of some sort. One might expect that the first stages of its becoming circular under the influence of surface tension would involve the formation of bulbous ends, giving a sort of dumbbell shape. Presumably, a thin enough rectangle would pass through a similar shape. Related behaviour might be expected at an outward-pointing cusp or corner, and by extension, a rounded cavity may form at an inward-pointing corner. Apparently, conditions at an inward-pointing cusp delicately avoid the latter. These speculations make it tempting to construct maps having bulbous ends from known ovals, such as the ovals of Part 1 or the elliptic body. For example, starting with (41), let $w = z/(c^2 - z^2)$, $c > (a+b)$. This leaves the w-image of $|\zeta| = (a/b)^{\frac{1}{2}}$ intact as a cut, neither opening a hole nor having overlaps, and provides plausible enough shapes. Given, however, the mathematical difficulty of avoiding a discontinuous $\psi(\zeta)$ along the continuum of points on the cut with just a few adjustable parameters, such exercises seem *a priori* likely to fail. Life is short, and one would want a reason to think a particular conjecture was especially hopeful.† More generally, attempts to guess an interesting map should be guided by physical intuition, by the constraints on the functional form of the mapping that seem inherent in this type of flow, and by the difficulties likely to be encountered in determining the parameters of the map. (See Part 1.) One must not expect too much: that shapes whose singular limit possesses corners or outward-pointing cusps apparently evolve according to maps that are hard to guess or uselessly cumbersome should be less surprising than there being interesting shapes that evolve with remarkable simplicity.

The type of flow considered is quite restricted. The problem, once reduced to dimensionless form is specified entirely by geometry: there are no variable physical parameters. It is doubtful that the formalism would be useful if there were. In particular, cases involving the following complications seem unlikely to yield to these methods: inertial effects, gravity, a superposed extensional flow (e.g. two cylinders coalescing by surface tension while simultaneously being elongated), or two viscosities (e.g. an infinite region of one liquid with a hole, as in §2, containing a second liquid). Also, given the simplicity of plane flow, one cannot be optimistic that axisymmetric three-dimensional flows would also admit a shape evolution describable by simple conformal maps. Perhaps the most immediately fruitful extension of the theory would be the development of an effective constructional method based on (3). What was said in Part 1 may be reiterated: the success of the description, though limited, is suggestive of deep connections between geometry and the dynamics of this class of problems, and insights into those connections may lead to methods more generally applicable to capillarity-driven flows.

Work performed under the auspices of the US Department of Energy by the Lawrence Livermore National Laboratory under contract W-7405-ENG-48.

REFERENCES

Gradshteyn, I. S. & Rhyzhik, I. M. 1980 *Table of Integrals, Series and Products* (transl. A. Jeffrey, corrected and enlarged edition). Academic.

Hopper, R. W. 1990 Plane strokes flow driven by capillarity on a free surface. *J. Fluid Mech.* **213**, 349–375 (referred to herein as Part 1).‡

Hopper, R. W. 1991 Plane Stokes flow driven by capillarity on the free surfaces of a doubly-connected region. *Lawrence Livermore Natl Lab. Internal Rep.* UCRL-ID-105872 (January 7, 1991).

Korn, G. A. & Korn, T. M. 1961 *Mathematical Handbook for Scientists and Engineers*, §1.8-3. McGraw-Hill.

Kuiken, H. K. 1990 Viscous sintering: the surface-tension-driven flow of a liquid form under the influence of curvature gradients at its surface. *J. Fluid Mech.* **214**, 503–515. §4.

Muskhelishvili, N. I. 1953 *Some Basic Problems in the Mathematical Theory of Elasticity* (transl. J. R. M. Radok). Groningen: P. Noordhoff.

Sokolnikov, I. S. 1956 *Mathematical Theory of Elasticity*, 2nd Edn. McGraw Hill.

† Actually, the bulbous ellipse is especially likely to fail. The $w(z)$ transformation is essentially that used in the coalescence of equal cylinders (Part 1), and it seems unlikely that a single function correctly gives flows that are the image of an arbitrary ellipse, not just a circle.

‡ The following errata are known: page 349, 'footnote on p. 000' should read 'footnote on p. 355'; page 357, 'footnote on p. 00' should read 'footnote on p. 353'; page 365, 'footnote on p. 00' should read 'footnote on p. 358'; page 353, equation (6), the integrand should be $[T_x(s)+\mathrm{i}T_y(s)]$; page 360, footnote, the notation in the first sentence is incorrect, but the right-hand side of the equation (the sum) correctly gives the analytic continuation of $\Omega(\sigma,t)^*$ onto $|\zeta| \leqslant 1$; page 363, equation (51), the h should be η.